1. Use Cauchy's residues theorem to evaluate the integral of each of these functions around the circle $|z|=3$ in the positive sense.

(a) $\dfrac{\exp(-z)}{z^2}$ (b) $\dfrac{\exp(-z)}{(z-1)^2}$ (c) $\dfrac{z+1}{z^2-2z}$

Solution:

In each part, C denotes the positively oriented circle $|z|=3$.

(a) To evaluate $\int_C \frac{\exp(-z)}{z^2}\,dz$, we need the residue of integrand at $z=0$. From the Laurent series.

$$\frac{\exp(-z)}{z^2} = \frac{1}{z^2}\left(1-\frac{z}{1!}+\frac{z^2}{2!}-\frac{z^3}{3!}+\cdots\right) = \frac{1}{z^2}-\frac{1}{1!}\frac{1}{z}+\frac{1}{2!}-\frac{z}{3!}+\cdots$$

$$(0<|z|<\infty)$$

we see that the required residues is -1. Thus

$$\int_C \frac{\exp(-z)}{z^2}\,dz = 2\pi i(-1) = -2\pi i$$

(b). The Taylor series for $\exp(-z)$ at $z=-1$ is.

$$\exp(-z) = e^{-1}+\frac{(-e^{-1})}{1!}(z-1)+\frac{e^{-1}}{2!}(z-1)^2+\cdots+\frac{(-1)^n e^{-1}}{n!}(z-1)^n+\cdots$$

Then $\frac{\exp(-z)}{(z-1)^2} = \frac{e^{-1}}{(z-1)^2} - \frac{e^{-1}}{z-1} + \frac{e^{-1}}{2!} - \frac{e^{-1}}{3!}(z-1) + \frac{e^{-1}}{4!}(z-1)^2 + \cdots$

Thus $\int_C \frac{\exp(-z)}{(z-1)^2}\, dz = 2\pi i\,(-e^{-1}) = -\frac{2\pi i}{e}$.

(c) As for the integral $\int_C \frac{z+1}{z^2-2z}\, dz$, we need the two residues of

$$\frac{z+1}{z^2-2z} = \frac{z+1}{z(z-2)}$$

one at $z=0$ and one at $z=2$.

The residues of $z=0$ can be found by writting

$$\frac{z+1}{z(z-2)} = \left(\frac{z+1}{z}\right)\left(\frac{1}{z-2}\right) = \left(-\frac{1}{2}\right)\left(1+\frac{1}{z}\right)\frac{1}{1-z/2}$$

$$= \left(-\frac{1}{2}\right)\left(1+\frac{1}{z}\right)\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\cdots\right) \qquad (0<|z|<2)$$

Observing that the coefficiente of $\frac{1}{z}$ is $-\frac{1}{2}$.

To obtain the residue at $z=2$, we write

$$\frac{z+1}{z(z-2)} = \frac{z-2+3}{z-2}\,\frac{1}{2+(z-2)} = \frac{1}{2}\left(1+\frac{3}{z-2}\right)\frac{1}{1+\left(\frac{z-2}{2}\right)}$$

$$= \frac{1}{2}\left(1+\frac{3}{z-2}\right)\left[1-\frac{z-2}{2}+\left(\frac{z-2}{2}\right)^2+\cdots\right], \quad 0<|z-2|<2$$

Note that the coefficient of $\frac{1}{z-2}$ in this produce is $\frac{3}{2}$.

By residue theorem $\int_C \frac{z+1}{z^2-2z}\, dz = 2\pi i\left(-\frac{1}{2}+\frac{3}{2}\right) = 2\pi i$

Theorem (P_{243} of textbook).

Let z_0 be an isolated singular point of a function f. The following two statements are quivalent:

(a) z_0 is a pole of order m $(m=1,2,\cdots)$ of f;

(b) $f(z)$ can be written in the form

$$f(z)=\frac{\phi(z)}{(z-z_0)^m} \qquad (m=1,2,\cdots)$$

where $\phi(z)$ is analytic and non zero at z_0.

Moreover, if statements (a) and (b) are true,

$$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0) \qquad \text{when } m=1$$

and

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \qquad \text{when } m=2,3,\cdots$$

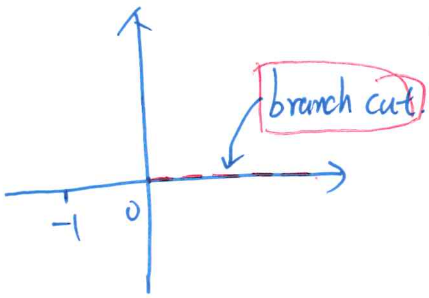
Q2. Find (a) $\displaystyle\operatorname*{Res}_{z=1} \frac{z^{1/4}}{z+1}$ $\quad(|z|>0,\ 0<\arg z<2\pi)$.

(b) $\displaystyle\operatorname*{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2}$

Solution: $f(z) = \dfrac{z^{1/4}}{z+1} = \dfrac{\phi(z)}{z+1}$,

where $\phi(z) = z^{1/4} = e^{\frac{1}{4}\log z}$ $\quad (|z|>0, \ 0<\arg z<2\pi)$

The function $\phi(z)$ is analytic throughout its domain of definition, indicated in the firgure below



Also $\phi(-1) = (-1)^{\frac{1}{4}}$

$= e^{\frac{1}{4}\log(-1)} = e^{\frac{1}{4}(\ln 1 + i\pi)}$

$= e^{\frac{\pi}{4}i} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$= \frac{1+i}{\sqrt{2}}$

Hence $\displaystyle \operatorname*{Res}_{z=-1} \frac{z^{1/4}}{z+1} = \phi(-1) = \frac{1+i}{\sqrt{2}}$

(b). $f(z) = \dfrac{\operatorname{Log} z}{(z^2+1)^2} = \dfrac{\phi(z)}{(z-i)^2}$

where $\phi(z) = \dfrac{\operatorname{Log} z}{(z+i)^2}$. It is clear that $f(z)$ has a pole of order 2 at $z=i$.

$$\operatorname*{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \phi'(i) = \frac{\pi+2i}{8}$$

Theorem (P 251) (we will use this theorem is Q3).

Let two functions p and q be analytic at a point z_0. If $p(z_0)\neq 0$, $q(z_0)=0$, and $q'(z_0)\neq 0$, then z_0 is a simple of the quotient $p(z)/q(z)$ and $\displaystyle \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$.

3. Let C_N denote the positively oriented boundary of the square whose edges lie along the lines

$$x = \pm (N+\tfrac{1}{2})\pi \quad \text{and} \quad y = \pm (N+\tfrac{1}{2})\pi$$

where N is a positive integer, Show that

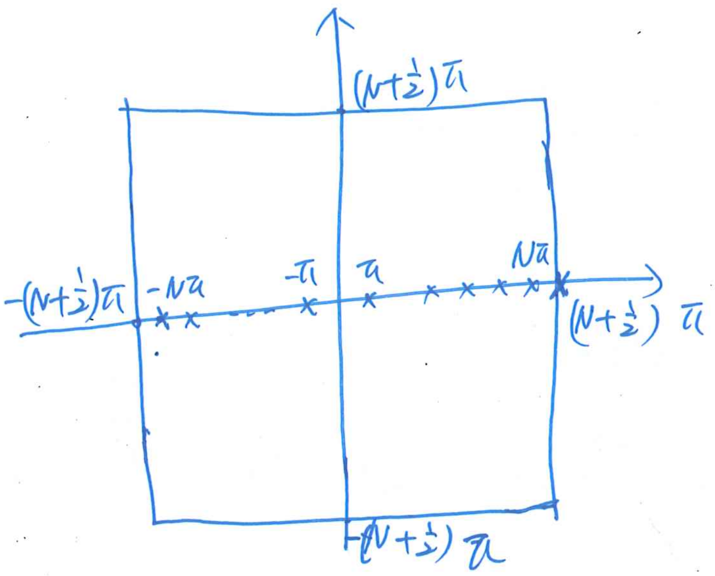
$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2 \sum_{n=1}^{N} \frac{(-1)^n}{n^2 \pi^2} \right]$$

Then, using the fact that the value of this integral tends to zero as N tends to infinity, point out how it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Solution:

The simple closed contour C_N is as shown in the firgure below



Within C_N, the function $f(z) = \frac{1}{z^2 \sin z}$ has isolated singularities are $z=0$ and $z = \pm n\pi$, $n=1,2,\cdots N$.

To find the residue at $z=0$, we recall the Laurent series

for $\frac{1}{\sin z} = \frac{1}{z} + \frac{1}{3!} z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right] z^3 + \cdots \qquad 0 < |z| < \pi.$

Then

$$\frac{1}{z^2 \sin z} = \frac{1}{z^2} \left\{ \frac{1}{z} + \frac{1}{3!} z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right] z^3 + \cdots \right\}$$

$$= \frac{1}{z^3} + \frac{1}{6}\frac{1}{z} + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right] z + \cdots \qquad (0 < |z| < \pi).$$

This tell us that $\frac{1}{z^2 \sin z}$ has a pole of order 3 at $z=0$

and $\underset{z=0}{\operatorname{Res}} \frac{1}{z^2 \sin z} = \frac{1}{6}$.

As for the points $z = \pm n\pi$ $(n=1, 2, \cdots N)$, write

$$\frac{1}{z^2 \sin z} = \frac{p(z)}{q(z)}, \quad \text{where } p(z) = 1 \text{ and } q(z) = z^2 \sin z$$

Since $p(\pm n\pi) = 1 \neq 0$, $q(\pm n\pi) = 0$ and

$$q'(\pm n\pi) = n^2\pi^2 \cos(n\pi) = (-1)^n n^2 \pi^2 \neq 0$$

it follows that

$$\underset{z=\pm n\pi}{\operatorname{Res}} \frac{1}{z^2 \sin z} = \frac{1}{(-1)^n n^2\pi^2} \frac{(-1)^n}{(-1)^n} = \frac{(-1)^n}{n^2\pi^2}$$

So, by residue theorem

$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2 \sum_{n=1}^{N} \frac{(-1)^n}{n^2 \pi^2} \right]$$

Rewriting this equation as

$$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} - \frac{\pi}{4i} \int_{C_N} \frac{dz}{z^2 \sin z}$$

then we use the fact that $\lim\limits_{N \to \infty} \int_{C_N} \frac{dz}{z^2 \sin z} = 0$. to

derive

$$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

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4. Consider the function $f(z) = \frac{1}{[q(z)]^2}$, where q is analytic at z_0, $q(z_0) = 0$ and $q'(z_0) \neq 0$. Show that z_0 is a pole of order $m=2$ of the function f, with residue $B_0 = -\frac{q''(z_0)}{[q'(z_0)]^3}$.

Solution:

We are given that $f(z) = 1/[q(z)]^2$, where q is analytic at z_0, $q(z_0) = 0$ and $q'(z_0) \neq 0$. These conditions on q tell us that q has a zero of order $m=1$ at z_0. Hence $q(z) = (z - z_0)\, g(z)$. where g is a function that is analytic and nonzero at z_0, and this enable us to write

$$f(z) = \frac{\phi(z)}{(z-z_0)^2} \qquad \text{where } \phi(z) = \frac{1}{[g(z)]^2}$$

so f has a pole of order 2 at z_0 and

$$\operatorname*{Res}_{z=z_0} f(z) = \phi'(z_0) = -\frac{2g'(z_0)}{[g(z_0)]^3}$$

Since $q(z) = (z-z_0)\,g(z)$, then

$$q'(z) = (z-z_0)\,g'(z) + g(z) \text{ and}$$

$$q''(z) = (z-z_0)\,g''(z) + 2g'(z)$$

Setting $z = z_0$ in above two equation yields

$$q'(z_0) = g(z_0) \qquad q''(z_0) = 2g'(z_0)$$

Consequently, our expression for the residue of f at z_0 can be put in the desired form:

$$\operatorname*{Res}_{z=0} f(z) = -\frac{q''(z_0)}{[q'(z_0)]^3}.$$